Classical tests of multidimensional gravity: negative result

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Abstract. In Kaluza-Klein model with toroidal extra dimensions, we obtain the metric coefficients in a weak field approximation for delta-shaped matter sources. These metric coefficients are applied to calculate the formulas for frequency shift, perihelion shift, deflection of light and parameterized post-Newtonian (PPN) parameters. In the leading order of approximation, the formula for frequency shift coincides with well known general relativity expression. However, for perihelion shift, light deflection and PPN parameter γ we obtain formulas $D\pi r_g/[(D-2)a(1-e^2)]$, $(D-1)r_g/[(D-2)\rho]$ and $1/(D-2)$ respectively, where D is a total number of spatial dimensions. These expressions demonstrate good agreement with experimental data only in the case of ordinary three-dimensional ($D=3$) space. This result does not depend on the size of the extra dimensions. Therefore, in considered multidimensional Kaluza-Klein models the point-like masses cannot produce gravitational field which corresponds to the classical gravitational tests.

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1. Introduction

The idea of the multidimensionality of our Universe demanded by the theories of unification of the fundamental interactions is one of the most breathtaking ideas of theoretical physics. It takes its origin from the pioneering papers by Th.Kaluza and O.Klein [1] and now the most self-consistent modern theories of unification such as superstrings, supergravity and M-theory are constructed in spacetime with extra dimensions [2]. Different aspects of the idea of the multidimensionality are intensively used in numerous modern articles. Therefore, it is very important to suggest experiments which can reveal the extra dimensions. For example, one of the aims of Large Hadronic Collider consists in detecting of Kaluza-Klein particles which correspond to excitations of the internal spaces (see e.g. [3]). On the other hand, if we can show that the existence of the extra dimensions is contrary to observations, then these theories are prohibited. This important problem is extensively discussed in recent scientific literature (see e.g. [4]-[10]).

It is well known that classical gravitational tests such as frequency shift, perihelion shift, deflection of light and time delay of radar echoes (the Shapiro time delay effect) are crucial tests of any gravitational theory. For example, there is the significant discrepancy for Mercury between the measurement value of the perihelion shift and its calculated value using Newton's formalism [11]. It indicates that non-relativistic Newton's theory of gravity is not complete. This problem was resolved with the help of general relativity which is in good agreement with observations. Similar situation happened with deflection of light [12]. The Shapiro time delay effect is used to get an upper limit for the parameterized post-Newtonian parameter γ [13]. Obviously, multidimensional gravitational theories should also be in concordance with these experimental data. To check it, the corresponding estimates were carried out in a number of papers. For example, in [8], it was investigated the well known multidimensional black hole solution [14] and the authors obtained a negative result. However, this result was clear from the very beginning because the solution [14] does not have non-relativistic Newtonian limit in the case of extra dimensions. Definitely, in solar system such solutions lead to results which are far from the experimental data. The 5-D soliton metrics [15]-[17] were explored in [4]-[7]. In papers [5] and [6], it was found the range of parameters for which classical gravitational tests for these metrics satisfy the observational values. The black string (see e.g. [18]) is a particular limiting case of such solutions with a trivial metric coefficient for the extra dimension. However, it can be easily shown that such solutions do not correspond to the point-like matter sources.

In 5-D non-factorizable brane world model, classical gravitational tests were investigated in [19]. Here, the model contains one free parameter associated with the bulk Weyl tensor. For appropriate values of this parameter, the perihelion shift in this model does not contradict observations. Certainly, this result is of interest and it is necessary to examine carefully this model to verify the naturalness of the conditions imposed.

In our paper we consider classical gravitational tests in Kaluza-Klein models (factorizable geometry) with an arbitrary number of spatial dimensions $D \geq 3$. We suppose that in the absence of gravitating masses the metric is a flat one. Gravitating point-like masses (moving or at rest) perturb this metric and we consider these perturbations in a weak field approximation. In this approximation, we obtain the asymptotic form of the metric coefficients. Then we admit that, first, the extra

dimensions are compact and have the topology of tori and, second, gravitational potential far away from gravitating masses tends to non-relativistic Newtonian limit. In the case of a gravitating mass at rest, the obtained metric coefficients are used to calculate frequency shift, perihelion shift, deflection of light and parameterized post-Newtonian (PPN) parameters. We demonstrate that for the frequency shift type experiment it is hardly possible to observe the difference between the usual four-dimensional general relativity and multidimensional Kaluza-Klein models. However, the situation is quite different for perihelion shift, deflection of light and PPN parameters. In these cases we get formulas which generalize the corresponding ones in general relativity. We show that formulas for perihelion shift, deflection of light and PPN parameter γ depend on a total number of spatial dimensions and they are in good agreement with observations only in ordinary three-dimensional space. It is important to note that this result does not depend explicitly on the size of the extra dimensions‡. So, we cannot avoid the problem with classical gravitational tests in a limit of arbitrary small (but non-zero!) size of the extra dimensions. It is worth noting that in paper [9] the authors arrived at the same conclusions in spite of they use the different approach.

Therefore, our results show that in considered multidimensional Kaluza-Klein models the point-like gravitating masses cannot produce gravitational field which corresponds to the classical gravitational tests.

The paper is structured as follows. In section 2 we get the asymptotic metric coefficients in the weak field limit for the delta-shaped matter gravitating source. These metric coefficients are applied to calculate the formulas of frequency shift, perihelion shift, deflection of light and PPN parameters in section 3. The main results are summarized in the concluding section 4.

2. Weak gravitational field approximation

To start with, we consider the general form of the multidimensional metric:

$$ds^2 = g_{ik}dx^i dx^k = g_{00}\left(dx^0\right)^2 + 2g_{0\alpha}dx^0 dx^\alpha + g_{\alpha\beta}dx^\alpha dx^\beta\,, \tag{2.1}$$

where the Latin indices $i, k = 0, 1, \dots, D$ and the Greek indices $\alpha, \beta = 1, \dots, D$. D is the total number of spatial dimensions. We make the natural assumption that in the case of the absence of matter sources the spacetime is Minkowski spacetime: $g_{00} = \eta_{00} = 1$, $g_{0\alpha} = \eta_{0\alpha} = 0$, $g_{\alpha\beta} = \eta_{\alpha\beta} = -\delta_{\alpha\beta}$. At the same time, the extra dimensions may have the topology of tori. In the presence of matter, the metric is not a Minkowskian one and we will investigate it in the weak field limit. This means that the gravitational field is weak and the velocities of the test bodies are small compared to the speed of light c. In this case the metric is only slightly perturbed from its flat spacetime value:

$$g_{ik} \approx \eta_{ik} + h_{ik}\,, \tag{2.2}$$

where h_{ik} are corrections of the order $1/c^2$. In particular, $h_{00} \equiv 2\varphi/c^2$. Later we will demonstrate that φ is the non-relativistic gravitational potential. The same conclusion with respect to φ can be easily obtained from the comparison of the non-relativistic action of a test mass moving in a gravitational field with its relativistic action. To

‡ In the leading order of approximation, our formulas do not depend on sizes of the extra dimensions. All correction terms, where the sizes of the extra dimensions appear, are exponentially suppressed.

get the other correction terms up to the same order $1/c^2$, we should consider the multidimensional Einstein equation

$$R_{ik} = \frac{2S_D\tilde{G}_{\mathcal{D}}}{c^4}\left(T_{ik} - \frac{1}{D-1}g_{ik}T\right), \tag{2.3}$$

where $S_D = 2\pi^{D/2}/\Gamma(D/2)$ is the total solid angle (surface area of the $(D-1)$-dimensional sphere of unit radius), $\tilde{G}_{\mathcal{D}}$ is the gravitational constant in the $(\mathcal{D} = D+1)$-dimensional spacetime. We are going to investigate the weak field approximation where gravitational field is generated by N moving point masses. Therefore, the energy-momentum tensor is

$$T^{ik} = \sum_{p=1}^{N} m_p \left[(-1)^D g\right]^{-1/2} \frac{dx^i}{dt}\frac{dx^k}{dt}\frac{cdt}{ds}\delta(\mathbf{r}-\mathbf{r}_p), \tag{2.4}$$

where m_p is the rest mass and $\mathbf{r}_p$ is the radius vector of the p-th particle respectively. All radius vectors $\mathbf{r}$ and $\mathbf{r}_p$ are D-dimensional, e.g. $\mathbf{r} = (x^1, x^2, \ldots, x^D)$ where x^α are coordinates in metric (2.1). The rest mass density is

$$\rho \equiv \sum_{p=1}^{N} m_p\, \delta(\mathbf{r}-\mathbf{r}_p). \tag{2.5}$$

2.1. $1/c^2$ correction terms

Obviously, to hold in the right hand side of (2.3) the terms up to the order $1/c^2$, the components of energy-momentum tensor (2.4) are approximated as

$$T_{00} \approx \rho c^2, \quad T_{0\alpha} \approx 0, \quad T_{\alpha\beta} \approx 0 \quad \Rightarrow \quad T = T_i^i \approx \rho c^2. \tag{2.6}$$

Taking into account that h_{ik} are of the order of $1/c^2$, the covariant components of the Riemann and Ricci tensors

$$R_{iklm} = \frac{1}{2}\left(\frac{\partial^2 g_{im}}{\partial x^k \partial x^l} + \frac{\partial^2 g_{kl}}{\partial x^i \partial x^m} - \frac{\partial^2 g_{il}}{\partial x^k \partial x^m} - \frac{\partial^2 g_{km}}{\partial x^i \partial x^l}\right) + g_{np}\left(\Gamma^n_{kl}\Gamma^p_{im} - \Gamma^n_{km}\Gamma^p_{il}\right), \quad R_{km} = g^{il}R_{iklm} \tag{2.7}$$

up to the same order read correspondingly:

$$R_{iklm} \approx \frac{1}{2}\left(\frac{\partial^2 h_{im}}{\partial x^k \partial x^l} + \frac{\partial^2 h_{kl}}{\partial x^i \partial x^m} - \frac{\partial^2 h_{il}}{\partial x^k \partial x^m} - \frac{\partial^2 h_{km}}{\partial x^i \partial x^l}\right), \tag{2.8}$$

$$\begin{aligned} R_{km} &\approx \frac{1}{2}\eta^{il}\left(\frac{\partial^2 h_{im}}{\partial x^k \partial x^l} + \frac{\partial^2 h_{kl}}{\partial x^i \partial x^m} - \frac{\partial^2 h_{il}}{\partial x^k \partial x^m} - \frac{\partial^2 h_{km}}{\partial x^i \partial x^l}\right) \\ &= \frac{1}{2}\left(\frac{\partial^2 h^l_m}{\partial x^k \partial x^l} + \frac{\partial^2 h^l_k}{\partial x^m \partial x^l} - \frac{\partial^2 h^l_l}{\partial x^k \partial x^m} - \eta^{il}\frac{\partial^2 h_{km}}{\partial x^i \partial x^l}\right), \end{aligned} \tag{2.9}$$

where $h^i_k \equiv \eta^{im}h_{mk}$. With the help of the gauge conditions

$$\frac{\partial}{\partial x^k}\left(h^k_i - \frac{1}{2}h^l_l\delta^k_i\right) = 0, \tag{2.10}$$

the formula (2.9) can be written in the form

$$R_{km} \approx -\frac{1}{2}\eta^{il}\frac{\partial^2 h_{km}}{\partial x^i \partial x^l}. \tag{2.11}$$

Taking into account that the derivatives with respect to $x^0 = ct$ are much smaller than the derivatives with respect to x^α, we obtain from (2.11):

$$R_{00} \approx -\frac{1}{2}\eta^{\alpha\beta}\frac{\partial^2 h_{00}}{\partial x^\alpha \partial x^\beta} = \frac{1}{2}\delta^{\alpha\beta}\frac{\partial^2 h_{00}}{\partial x^\alpha \partial x^\beta} = \frac{1}{2}\triangle h_{00}, \tag{2.12}$$

$$R_{0\alpha} \approx \frac{1}{2}\triangle h_{0\alpha}, \quad R_{\alpha\beta} \approx \frac{1}{2}\triangle h_{\alpha\beta} \tag{2.13}$$

where $\triangle = \delta^{\alpha\beta}\partial^2/\partial x^\alpha \partial x^\beta$ is the D-dimensional Laplace operator. It is worth noting that for the condition (2.10) up to the order $1/c^2$ holds

$$\frac{\partial}{\partial x^\beta}\left(h_\alpha^\beta - \frac{1}{2}h_l^l \delta_\alpha^\beta\right) = 0 + O(1/c^3)\,, \qquad \frac{\partial h_0^\beta}{\partial x^\beta} = 0 + O(1/c^3)\,. \tag{2.14}$$

Therefore, keeping in the left hand and right hand sides of (2.3) terms up to the order $1/c^2$ we obtain the following equations:

$$\begin{aligned} \triangle h_{00} &= \frac{2S_D G_{\mathcal{D}}}{c^2}\rho\,, \quad \triangle h_{0\alpha} = 0\,, \\ \triangle h_{\alpha\beta} &= \frac{1}{D-2}\cdot\frac{2S_D G_{\mathcal{D}}}{c^2}\rho\delta_{\alpha\beta}\,, \end{aligned} \tag{2.15}$$

where $G_{\mathcal{D}} = [2(D-2)/(D-1)]\,\tilde{G}_{\mathcal{D}}$. Substitution of $h_{00} = 2\varphi/c^2$ into the above equation for h_{00} demonstrates that φ satisfies the D-dimensional Poisson equation:

$$\triangle\varphi = S_D G_{\mathcal{D}}\rho\,. \tag{2.16}$$

Therefore, φ is the non-relativistic gravitational potential. From (2.15) we obtain

$$h_{0\alpha} = 0\,, \quad h_{\alpha\beta} = \frac{1}{D-2}\,h_{00}\delta_{\alpha\beta} = \frac{1}{D-2}\,\frac{2\varphi}{c^2}\delta_{\alpha\beta}\,. \tag{2.17}$$

It can be easily seen that in this approximation spacial coordinates of the metric (2.1) are the isotropic ones, i.e. the spacial part of the metric is conformally related to the Euclidean one. It is worth noting also that the relation $h_{\alpha\beta}/h_{00} = [1/(D-2)]\delta_{\alpha\beta}$ can be also obtained from the corresponding equations in papers [14, 20].

2.2. $1/c^3$ *and* $1/c^4$ *correction terms*

Now, we want to keep in metric (2.1) the terms up to the order $1/c^2$. Because the coordinate $x^0 = ct$ contains c, this means that in g_{00} and $g_{0\alpha}$ we should keep correction terms up to the order $1/c^4$ and $1/c^3$ respectively and to leave $g_{\alpha\beta}$ without changes in the form $g_{\alpha\beta} \approx \eta_{\alpha\beta} + h_{\alpha\beta}$ with $h_{\alpha\beta}$ from (2.17).

First, we investigate the energy-momentum tensor (2.4) which we split into three expressions:

$$T^{00} = \sum_{p=1}^{N} m_p c^2 [(-1)^D g]^{-1/2}\frac{cdt}{ds}\delta(\mathbf{r} - \mathbf{r}_p)\,, \tag{2.18}$$

$$T^{0\alpha} = \sum_{p=1}^{N} m_p c [(-1)^D g]^{-1/2} v_p^\alpha \frac{cdt}{ds}\delta(\mathbf{r} - \mathbf{r}_p)\,, \tag{2.19}$$

$$T^{\alpha\beta} = \sum_{p=1}^{N} m_p [(-1)^D g]^{-1/2} v_p^\alpha v_p^\beta \frac{cdt}{ds}\delta(\mathbf{r} - \mathbf{r}_p)\,, \tag{2.20}$$

where $v_p^\alpha = dx_p^\alpha/dt$. From (2.20) we obtain up to order 1 (in units c) the covariant components

$$T_{\alpha\beta} \approx \sum_{p=1}^{N} m_p v_{p\alpha} v_{p\beta} \delta(\mathbf{r} - \mathbf{r}_p) \,. \tag{2.21}$$

Thus, taking into account the prefactor $1/c^4$ in the right hand side of (2.3), these components can contribute to $g_{\alpha\beta}$ terms of the order of $1/c^4$ which is not of interest for us. For $T_{0\alpha}$ we find from (2.19):

$$T_{0\alpha} \approx -\sum_{p=1}^{N} m_p c v_{p\alpha} \delta(\mathbf{r} - \mathbf{r}_p) \,. \tag{2.22}$$

Hence, these components can give in $g_{0\alpha}$ terms of the order of $1/c^3$ which is of interest for us. Finally, for T_{00} we get from (2.18):

$$\begin{aligned} &T_{00} = g_{0i} g_{0k} T^{ik} \approx (g_{00})^2 T^{00} \\ &\approx \left(1 + \frac{2\varphi}{c^2}\right)^2 \left[(-1)^D \left(1 + \frac{2\varphi}{c^2}\right)\left(-1 + \frac{1}{D-2}\frac{2\varphi}{c^2}\right)^D\right]^{-1/2} \\ &\times \sum_{p=1}^{N} m_p c^2 \left[\left(1 + \frac{2\varphi}{c^2}\right) - \frac{v_p^2}{c^2}\right]^{-1/2} \delta(\mathbf{r} - \mathbf{r}_p) \approx \left(1 + \frac{4\varphi}{c^2}\right)\left(1 + \frac{1}{D-2}\frac{2\varphi}{c^2}\right) \\ &\times \sum_{p=1}^{N} m_p c^2 \left(1 - \frac{\varphi}{c^2} + \frac{v_p^2}{2c^2}\right)\delta(\mathbf{r} - \mathbf{r}_p) \approx \sum_{p=1}^{N} m_p c^2 \left(1 + \frac{3D-4}{D-2}\frac{\varphi}{c^2} + \frac{v_p^2}{2c^2}\right)\delta(\mathbf{r} - \mathbf{r}_p) \\ &= \sum_{p=1}^{N} m_p c^2 \delta(\mathbf{r} - \mathbf{r}_p) + \sum_{p=1}^{N} m_p \left(\frac{3D-4}{D-2}\varphi_p + \frac{1}{2} v_p^2\right)\delta(\mathbf{r} - \mathbf{r}_p) \,, \end{aligned} \tag{2.23}$$

where φ_p is potential of gravitational field in a point with radius vector $\mathbf{r}_p$. At the moment, we do not care about the fact that φ_p contains the infinite contribution of the p-th particle. Thus, up to order 1 we get

$$\begin{aligned} T = g^{ik} T_{ik} \approx g^{00} T_{00} + g^{\alpha\beta} T_{\alpha\beta} &\approx \sum_{p=1}^{N} m_p c^2 \delta(\mathbf{r} - \mathbf{r}_p) \\ &+ \sum_{p=1}^{N} m_p \left(\frac{D}{D-2}\varphi_p - \frac{1}{2} v_p^2\right)\delta(\mathbf{r} - \mathbf{r}_p) \,. \end{aligned} \tag{2.24}$$

With the help of (2.23) and (2.24) we obtain up to the order $1/c^4$:

$$\begin{aligned} \frac{2 S_D \tilde{G}_{\mathcal{D}}}{c^4}\left(T_{00} - \frac{1}{D-1} g_{00} T\right) &\approx \frac{S_D G_{\mathcal{D}}}{c^2} \sum_{p=1}^{N} m_p \delta(\mathbf{r} - \mathbf{r}_p) \\ + \frac{S_D G_{\mathcal{D}}}{c^4} \sum_{p=1}^{N} m_p &\left(\frac{3D-4}{D-2}\varphi_p + \frac{D}{2(D-2)} v_p^2\right)\delta(\mathbf{r} - \mathbf{r}_p). \end{aligned} \tag{2.25}$$

Similarly, from (2.22) and (2.24) we get up to the order $1/c^3$:

$$\frac{2 S_D \tilde{G}_{\mathcal{D}}}{c^4}\left(T_{0\alpha} - \frac{1}{D-1} g_{0\alpha} T\right) \approx -\frac{D-1}{D-2}\frac{S_D G_{\mathcal{D}}}{c^3} \sum_{p=1}^{N} m_p v_{p\alpha} \delta(\mathbf{r} - \mathbf{r}_p) \,. \tag{2.26}$$

Now, we shall work out the left hand side of (2.3) up to an appropriate orders of $1/c$. As we wrote above, we are looking for corrections of the order of $1/c^4$ and $1/c^3$ to the metric components g_{00} and $g_{0\alpha}$, respectively. To this end, it is convenient to present g_{ik} as follows:

$$g_{ik} \approx \eta_{ik} + h_{ik} + f_{ik}\,, \tag{2.27}$$

where f_{00} and $f_{0\alpha}$ are of the order of $1/c^4$ and $1/c^3$, respectively. Then, the Riemann curvature tensor (2.7) reads

$$\begin{aligned}
&R_{iklm} \approx \frac{1}{2}\left(\frac{\partial^2 h_{im}}{\partial x^k \partial x^l} + \frac{\partial^2 h_{kl}}{\partial x^i \partial x^m} - \frac{\partial^2 h_{il}}{\partial x^k \partial x^m} - \frac{\partial^2 h_{km}}{\partial x^i \partial x^l}\right)\\
&+\frac{1}{2}\left(\frac{\partial^2 f_{im}}{\partial x^k \partial x^l} + \frac{\partial^2 f_{kl}}{\partial x^i \partial x^m} - \frac{\partial^2 f_{il}}{\partial x^k \partial x^m} - \frac{\partial^2 f_{km}}{\partial x^i \partial x^l}\right) + \eta^{np}\left(\Gamma_{n,kl}\Gamma_{p,im} - \Gamma_{n,km}\Gamma_{p,il}\right)\\
&\approx \frac{1}{2}\left(\frac{\partial^2 h_{im}}{\partial x^k \partial x^l} + \frac{\partial^2 h_{kl}}{\partial x^i \partial x^m} - \frac{\partial^2 h_{il}}{\partial x^k \partial x^m} - \frac{\partial^2 h_{km}}{\partial x^i \partial x^l}\right)\\
&+\frac{1}{2}\left(\frac{\partial^2 f_{im}}{\partial x^k \partial x^l} + \frac{\partial^2 f_{kl}}{\partial x^i \partial x^m} - \frac{\partial^2 f_{il}}{\partial x^k \partial x^m} - \frac{\partial^2 f_{km}}{\partial x^i \partial x^l}\right)\\
&+\frac{1}{4}\eta^{np}\left(\frac{\partial h_{nk}}{\partial x^l} + \frac{\partial h_{nl}}{\partial x^k} - \frac{\partial h_{kl}}{\partial x^n}\right)\left(\frac{\partial h_{pi}}{\partial x^m} + \frac{\partial h_{pm}}{\partial x^i} - \frac{\partial h_{im}}{\partial x^p}\right)\\
&-\frac{1}{4}\eta^{np}\left(\frac{\partial h_{nk}}{\partial x^m} + \frac{\partial h_{nm}}{\partial x^k} - \frac{\partial h_{km}}{\partial x^n}\right)\left(\frac{\partial h_{pi}}{\partial x^l} + \frac{\partial h_{pl}}{\partial x^i} - \frac{\partial h_{il}}{\partial x^p}\right)\,.
\end{aligned} \tag{2.28}$$

From this formula we obtain the Ricci tensor:

$$\begin{aligned}
R_{km} \approx\; & -\frac{1}{2}\eta^{il}\frac{\partial^2 h_{km}}{\partial x^i \partial x^l} + \frac{1}{2}H_{km} - \frac{1}{2}\eta^{il}\frac{\partial^2 f_{km}}{\partial x^i \partial x^l} + \frac{1}{2}F_{km}\\
& -\frac{1}{2}\eta^{ij}\eta^{lp}h_{jp}\left(\frac{\partial^2 h_{im}}{\partial x^k \partial x^l} + \frac{\partial^2 h_{kl}}{\partial x^i \partial x^m} - \frac{\partial^2 h_{il}}{\partial x^k \partial x^m} - \frac{\partial^2 h_{km}}{\partial x^i \partial x^l}\right)\\
& +\frac{1}{4}\eta^{il}\eta^{np}\left(\frac{\partial h_{nk}}{\partial x^l} + \frac{\partial h_{nl}}{\partial x^k} - \frac{\partial h_{kl}}{\partial x^n}\right)\left(\frac{\partial h_{pi}}{\partial x^m} + \frac{\partial h_{pm}}{\partial x^i} - \frac{\partial h_{im}}{\partial x^p}\right)\\
& -\frac{1}{4}\eta^{il}\eta^{np}\left(\frac{\partial h_{nk}}{\partial x^m} + \frac{\partial h_{nm}}{\partial x^k} - \frac{\partial h_{km}}{\partial x^n}\right)\left(\frac{\partial h_{pi}}{\partial x^l} + \frac{\partial h_{pl}}{\partial x^i} - \frac{\partial h_{il}}{\partial x^p}\right)\,,
\end{aligned} \tag{2.29}$$

where we introduced the notations:

$$\begin{aligned}
H_{km} &= \eta^{il}\left(\frac{\partial^2 h_{im}}{\partial x^k \partial x^l} + \frac{\partial^2 h_{kl}}{\partial x^i \partial x^m} - \frac{\partial^2 h_{il}}{\partial x^k \partial x^m}\right)\\
&= \frac{\partial^2 h^0_m}{\partial x^k \partial x^0} + \frac{\partial^2 h^0_k}{\partial x^0 \partial x^m} + \frac{\partial^2 h^\beta_m}{\partial x^k \partial x^\beta} + \frac{\partial^2 h^\beta_k}{\partial x^\beta \partial x^m} - \frac{\partial^2 h^l_l}{\partial x^k \partial x^m}
\end{aligned} \tag{2.30}$$

and

$$\begin{aligned}
F_{km} &= \eta^{il}\left(\frac{\partial^2 f_{im}}{\partial x^k \partial x^l} + \frac{\partial^2 f_{kl}}{\partial x^i \partial x^m} - \frac{\partial^2 f_{il}}{\partial x^k \partial x^m}\right)\\
&= \frac{\partial^2 f^0_m}{\partial x^k \partial x^0} + \frac{\partial^2 f^0_k}{\partial x^0 \partial x^m} + \frac{\partial^2 f^\beta_m}{\partial x^k \partial x^\beta} + \frac{\partial^2 f^\beta_k}{\partial x^\beta \partial x^m} - \frac{\partial^2 f^l_l}{\partial x^k \partial x^m}\,.
\end{aligned} \tag{2.31}$$

Taking into account that $h^\alpha_0 = h^0_\alpha \equiv 0$, we get

$$H_{00} = \frac{\partial^2 h^0_0}{\partial (x^0)^2} - \frac{\partial^2 h^\alpha_\alpha}{\partial (x^0)^2} \tag{2.32}$$

and

$$H_{0\alpha} = \frac{\partial^2 h^\beta_\alpha}{\partial x^0 \partial x^\beta} - \frac{\partial^2 h^\beta_\beta}{\partial x^0 \partial x^\alpha}\,, \tag{2.33}$$

which are of the order $1/c^4$ and $1/c^3$, respectively. For the components F_{km} we obtain

$$F_{00} \approx 2\frac{\partial^2 f^\beta_0}{\partial x^0 \partial x^\beta} = \frac{\partial^2 h^\beta_\beta}{\partial (x^0)^2}\,, \quad F_{0\alpha} \approx \frac{\partial^2 f^\beta_0}{\partial x^\beta \partial x^\alpha} = \frac{1}{2}\frac{\partial^2 h^\beta_\beta}{\partial x^0 \partial x^\alpha}\,, \tag{2.34}$$

which are defined up to the orders $1/c^4$ and $1/c^3$, respectively. To get these expressions, we use the gauge condition

$$\frac{\partial f^\beta_0}{\partial x^\beta} = \frac{1}{2}\frac{\partial h^\beta_\beta}{\partial x^0}\,. \tag{2.35}$$

Therefore, the 00-component of the Ricci tensor reads

$$\begin{aligned} R_{00} \approx\; & \frac{1}{2}\triangle h_{00} - \frac{1}{2}\frac{\partial^2 h_{00}}{\partial (x^0)^2} + \frac{1}{2}\frac{\partial^2 h^0_0}{\partial (x^0)^2} - \frac{1}{2}\frac{\partial^2 h^\alpha_\alpha}{\partial (x^0)^2} \\ & + \frac{1}{2}\triangle f_{00} + \frac{1}{2}\frac{\partial^2 h^\beta_\beta}{\partial (x^0)^2} + \frac{1}{2}\eta^{ij}\eta^{lp}h_{jp}\frac{\partial^2 h_{00}}{\partial x^i \partial x^l} \\ & + \frac{1}{4}\eta^{il}\eta^{np}\left(\frac{\partial h_{n0}}{\partial x^l} - \frac{\partial h_{0l}}{\partial x^n}\right)\left(\frac{\partial h_{p0}}{\partial x^i} - \frac{\partial h_{i0}}{\partial x^p}\right) \\ & - \frac{1}{4}\eta^{il}\eta^{np}\left(-\frac{\partial h_{00}}{\partial x^n}\right)\left(\frac{\partial h_{pi}}{\partial x^l} + \frac{\partial h_{pl}}{\partial x^i} - \frac{\partial h_{il}}{\partial x^p}\right)\,. \end{aligned} \tag{2.36}$$

With the help of the following relations (which are correct up to the order $1/c^4$):

$$\frac{1}{2}\eta^{ij}\eta^{lp}h_{jp}\frac{\partial^2 h_{00}}{\partial x^i \partial x^l} \approx \frac{1}{2}h_{11}\triangle h_{00} = \frac{1}{D-2}\cdot\frac{2}{c^4}\varphi\triangle\varphi\,, \tag{2.37}$$

$$\frac{1}{4}\eta^{il}\eta^{np}\left(\frac{\partial h_{n0}}{\partial x^l} - \frac{\partial h_{0l}}{\partial x^n}\right)\left(\frac{\partial h_{p0}}{\partial x^i} - \frac{\partial h_{i0}}{\partial x^p}\right) \approx \frac{1}{2}\eta^{il}\frac{\partial h_{00}}{\partial x^l}\frac{\partial h_{00}}{\partial x^i} \approx -\frac{2}{c^4}(\nabla\varphi)^2 \tag{2.38}$$

and

$$\eta^{il}\eta^{np}\frac{\partial h_{00}}{\partial x^n}\left(\frac{\partial h_{pi}}{\partial x^l} + \frac{\partial h_{pl}}{\partial x^i} - \frac{\partial h_{il}}{\partial x^p}\right) \approx \frac{\partial h_{00}}{\partial x^\beta}\frac{\partial}{\partial x^\alpha}\left(2h^{\alpha\beta} - \eta^{\alpha\beta}h^l_l\right) \approx 0\,, \tag{2.39}$$

where the condition (2.14) was used in the latter expression, the 00-component (2.36) of the Ricci tensor takes the form

$$R_{00} \approx \frac{1}{c^2}\triangle\varphi + \frac{1}{2}\triangle f_{00} + \frac{1}{D-2}\cdot\frac{2}{c^4}\varphi\triangle\varphi - \frac{2}{c^4}(\nabla\varphi)^2\,. \tag{2.40}$$

The 0α component of the Ricci tensor (2.29) up to the order $1/c^3$ reads

$$R_{0\alpha} \approx -\frac{1}{2}\eta^{il}\frac{\partial^2 h_{0\alpha}}{\partial x^i \partial x^l} + \frac{1}{2}H_{0\alpha} - \frac{1}{2}\eta^{il}\frac{\partial^2 f_{0\alpha}}{\partial x^i \partial x^l} + \frac{1}{2}F_{0\alpha} \approx \frac{1}{2}\triangle f_{0\alpha} + \frac{1}{2c^3}\frac{\partial^2\varphi}{\partial t\partial x^\alpha}\,, \tag{2.41}$$

where we used formulas (2.33) and (2.34).

Now, we come back to Einstein equation (2.3). Substituting (2.25) and (2.40) into (2.3) and taking into account (2.5) and (2.16), we get the following equation for f_{00}:

$$\begin{aligned} & \triangle f_{00} + \frac{1}{D-2}\,\frac{4}{c^4}\varphi\triangle\varphi - \frac{4}{c^4}(\nabla\varphi)^2 \\ & = \frac{2S_D G_{\mathcal{D}}}{c^4}\sum_{p=1}^N m_p\left(\frac{3D-4}{D-2}\varphi_p + \frac{D}{2(D-2)}v_p^2\right)\delta(\mathbf{r}-\mathbf{r}_p)\,. \end{aligned} \tag{2.42}$$

With the help of the auxiliary equation:

$$4(\nabla\varphi)^2 = 2\triangle(\varphi^2) - 4\varphi\triangle\varphi \tag{2.43}$$

and equations (2.5) and (2.16), equation (2.42) takes the form:

$$\triangle\left(f_{00} - \frac{2}{c^4}\varphi^2\right) = \frac{2S_D G_{\mathcal{D}}}{c^4}\sum_{p=1}^{N} m_p\left(\varphi_p' + \frac{D}{2(D-2)}v_p^2\right)\delta(\mathbf{r}-\mathbf{r}_p)\,. \tag{2.44}$$

Here, φ_p' is the potential of the gravitational field in a point with radius vector $\mathbf{r}_p$ produced by all particles, except of the p-th. Substraction of the infinite contribution of the gravitational field of the p-th particle corresponds to a renormalization of its mass (see [21]). The solution of (2.44) is:

$$f_{00} = \frac{2}{c^4}\varphi^2(\mathbf{r}) + \frac{2}{c^4}\sum_{p=1}^{N}\varphi_p'\varphi'(\mathbf{r}-\mathbf{r}_p) + \frac{D}{D-2}\cdot\frac{1}{c^4}\sum_{p=1}^{N}v_p^2\varphi'(\mathbf{r}-\mathbf{r}_p)\,, \tag{2.45}$$

where $\varphi'(\mathbf{r}-\mathbf{r}_p)$ is the potential of the gravitational field of the p-th particle which satisfies the Poisson equation:

$$\triangle\varphi' = \delta^{\alpha\beta}\frac{\partial^2\varphi'}{\partial x^\alpha\partial x^\beta} = S_D G_{\mathcal{D}} m_p\delta(\mathbf{r}-\mathbf{r}_p)\,. \tag{2.46}$$

It can be easily verified with the help of (2.5) and (2.16) that $\varphi'(\mathbf{r}-\mathbf{r}_p)$ satisfies the condition:

$$\varphi(\mathbf{r}) = \sum_{p=1}^{N}\varphi'(\mathbf{r}-\mathbf{r}_p)\,. \tag{2.47}$$

Therefore, substituting $h_{00} = 2\varphi/c^2$ and f_{00} into (2.27), we obtain g_{00} up to the order $1/c^4$:

$$\begin{aligned} g_{00} \approx{}& 1 + \frac{2}{c^2}\varphi(\mathbf{r}) + \frac{2}{c^4}\varphi^2(\mathbf{r}) \\ &+ \frac{2}{c^4}\sum_{p=1}^{N}\varphi_p'\varphi'(\mathbf{r}-\mathbf{r}_p) + \frac{D}{D-2}\cdot\frac{1}{c^4}\sum_{p=1}^{N}v_p^2\varphi'(\mathbf{r}-\mathbf{r}_p)\,. \end{aligned} \tag{2.48}$$

We should mention that the radius vectors $\mathbf{r}_p$ of the moving gravitating masses depend on time. In this case, potential $\varphi(\mathbf{r})$ in (2.47) also depends on time.

The equation for $f_{0\alpha}$ can be obtained by substitution of (2.26) and (2.41) into Einstein equation (2.3):

$$\triangle f_{0\alpha} = -\frac{2(D-1)}{D-2}\frac{S_D G_{\mathcal{D}}}{c^3}\sum_{p=1}^{N} m_p v_{p\alpha}\delta(\mathbf{r}-\mathbf{r}_p) - \frac{1}{c^3}\frac{\partial^2\varphi}{\partial t\partial x^\alpha}, \tag{2.49}$$

whose solution is:

$$f_{0\alpha} = -\frac{2(D-1)}{D-2}\cdot\frac{1}{c^3}\sum_{p=1}^{N}v_{p\alpha}\varphi'(\mathbf{r}-\mathbf{r}_p) - \frac{1}{c^3}\frac{\partial^2 f}{\partial t\partial x^\alpha}\,, \tag{2.50}$$

where the function f satisfies equation

$$\triangle f = \delta^{\alpha\beta}\frac{\partial^2 f}{\partial x^\alpha\partial x^\beta} = \varphi(\mathbf{r})\,. \tag{2.51}$$

Therefore, substituting $h_{0\alpha} = 0$ and $f_{0\alpha}$ into (2.27), we get $g_{0\alpha}$ up to the order $1/c^3$:

$$g_{0\alpha} \approx -\frac{2(D-1)}{D-2}\frac{1}{c^3}\sum_{p=1}^{N} v_{p\alpha}\varphi'(\mathbf{r}-\mathbf{r}_p) - \frac{1}{c^3}\frac{\partial^2 f}{\partial t \partial x^\alpha}\,. \tag{2.52}$$

It is necessary to note that in the three-dimensional case $D = 3$ (2.48) and (2.52) exactly coincide with (106.13) and (106.14) in [21] if we take into account that $\varphi'(\mathbf{r}-\mathbf{r}_p) = -G_N m_p/|\mathbf{r}-\mathbf{r}_p|$.

From now on we shall consider the case of one gravitating particle of mass $m_1 \equiv m$ at rest in our 3-D space but, for generality, moving with constant speed in extra dimensions. That is $p = 1 \Rightarrow \varphi'_1 = 0$ and $v^\alpha = dx^\alpha/dt = (0, 0, 0, v_4, v_5, ..., v_D)$, where $v_4, v_5, ..., v_D$ are constants. In this case (2.48) and (2.52) are reduced correspondingly to

$$g_{00} \approx 1 + \frac{2}{c^2}\varphi(\mathbf{r}) + \frac{2}{c^4}\varphi^2(\mathbf{r}) + \frac{Dv^2}{(D-2)c^4}\varphi(\mathbf{r}) \tag{2.53}$$

and

$$g_{0\alpha} \approx -\frac{2(D-1)v_\alpha}{(D-2)c^3}\varphi(\mathbf{r}) - \frac{1}{c^3}\frac{\partial^2 f}{\partial t \partial x^\alpha}\,, \tag{2.54}$$

where $\varphi(\mathbf{r})$ satisfies the Poisson equation:

$$\triangle\varphi = \delta^{\alpha\beta}\frac{\partial^2\varphi}{\partial x^\alpha \partial x^\beta} = S_D G_{\mathcal{D}} m \delta(\mathbf{r}) \tag{2.55}$$

and $v^2 = -g_{\alpha\beta}v^\alpha v^\beta = v_4^2 + v_5^2 + ... + v_D^2 + O(1/c^2)$ (at the same accuracy $v_\beta = -v^\beta$). Obviously, the transition to the case where the gravitating mass is at rest both in our three-dimensional space and in the extra dimensions corresponds to the limit $v_\alpha = 0 \Rightarrow v^2 = 0$. In this case the potential $\varphi(\mathbf{r})$ as well as the function f do not depend on time t. We remind that the covariant components $g_{\alpha\beta}$ read (see (2.17)):

$$g_{\alpha\beta} \approx -\left(1 - \frac{1}{D-2}\cdot\frac{2}{c^2}\varphi(\mathbf{r})\right)\delta_{\alpha\beta}\,. \tag{2.56}$$

To get all above results, we did not use any concrete form of topology. The only things we used were assumptions of the flatness of metric in the absence of the gravitating masses and the weakness of the gravitational field and velocities of gravitating masses which perturb the flat metric. Now, to solve (2.55) we should specify the topology of space and the boundary conditions. We suppose that the $(D = 3 + d)$-dimensional space has the factorizable geometry of a product manifold $M_D = \mathbb{R}^3 \times T^d$. $\mathbb{R}^3$ describes the three-dimensional flat external (our) space and T^d is a torus which corresponds to a d-dimensional internal space with volume V_d. For this topology, and with the boundary condition that at infinitely large distances from the gravitating body the potential must go to the Newtonian expression, we can find the exact solution of the Poisson equation (2.55) [22, 23]. The boundary condition requires that the multidimensional $G_{\mathcal{D}}$ and Newtonian G_N gravitational constants are connected by the following condition: $S_D G_{\mathcal{D}}/V_d = 4\pi G_N$. Assuming that we consider the gravitational field of a gravitating mass m at distances much greater than periods of tori, we can restrict ourselves to the zero Kaluza-Klein mode. For example, this approximation is very well satisfied for the planets of the solar system because the inverse-square law experiments show that the extra dimensions in Kaluza-Klein models should not exceed submillimeter scales [24] (see however [22, 23]

for models with smeared extra dimensions where Newton's law preserves its shape for arbitrary distances). Then, the gravitational potential reads

$$\varphi(\mathbf{r}) \approx -\frac{G_N m}{r_3} = -\frac{r_g c^2}{2r_3}\,, \tag{2.57}$$

where r_3 is the length of a radius vector in three-dimensional space and we introduce three-dimensional Schwarzschild radius $r_g = 2G_N m/c^2$. As we mentioned above, the gravitating mass m is at rest in our three-dimensional space but may move in the extra dimensions. In this case, the extra dimensional components of D-dimensional radius vector of the gravitating particle depend on time. The exact formulas for the non-relativistic gravitational potential (see [22, 23]) show that this dependence "nests" only in non-zero Kaluza-Klein modes which are exponentially suppressed in considered approximation. Therefore, in this approximation potential $\varphi(\mathbf{r})$ in (2.57) does not depend on time.

It is worth noting that all the previous analysis works also in the case when the gravitating masses are uniformly smeared over some or all extra dimensions. Let us take for simplicity one ($p = 1$) gravitating mass $m_1 \equiv m$ which is smeared over all extra dimensions. Obviously, this mass can move only in our usual three dimensions: $v_1^\alpha = dx_1^\alpha/dt = (v_1^1, v_1^2, v_1^3, 0, \ldots, 0)$ and its rest mass density (2.5) now reads:

$$\rho = \left(m/\prod_{\alpha=1}^{d} a_\alpha\right)\delta(\mathbf{r}_3 - \mathbf{r}_{(1)3})\,, \tag{2.58}$$

where a_α are periods of tori. Then, the solution of the Poisson equation (2.16) exactly coincides with the Newton potential if the multidimensional $G_{\mathcal{D}}$ and Newtonian G_N gravitational constants are connected as $S_D G_{\mathcal{D}}/\prod_{\alpha=1}^{d} a_\alpha = 4\pi G_N$ [22, 23]. Therefore, in this case the approximate formula (2.57) becomes the exact equality:

$$\varphi(\mathbf{r}) = \varphi(\mathbf{r}_3) = -\frac{G_N m}{r_3} = -\frac{r_g c^2}{2r_3}\,. \tag{2.59}$$

In the approximation (2.57) (or with (2.59) for "smeared" extra dimensions), the covariant components (2.53), (2.54) and (2.56) take the form

$$\begin{aligned} g_{00} &\approx 1 - \frac{r_g}{r_3} + \frac{r_g^2}{2r_3^2} - \frac{Dv^2}{2(D-2)c^2}\frac{r_g}{r_3}\,,\\ g_{0\alpha} &\approx \frac{(D-1)v_\alpha}{(D-2)c}\frac{r_g}{r_3}, \quad g_{\alpha\beta} \approx -\left(1 + \frac{1}{D-2}\cdot\frac{r_g}{r_3}\right)\delta_{\alpha\beta}\,. \end{aligned} \tag{2.60}$$

For the contravariant components we obtain:

$$\begin{aligned} g^{00} &\approx 1 + \frac{r_g}{r_3} + \frac{r_g^2}{2r_3^2} + \frac{Dv^2}{2(D-2)c^2}\frac{r_g}{r_3}\,,\\ g^{0\alpha} &\approx -\frac{(D-1)v^\alpha}{(D-2)c}\frac{r_g}{r_3}, \quad g^{\alpha\beta} \approx -\left(1 - \frac{1}{D-2}\cdot\frac{r_g}{r_3}\right)\delta_{\alpha\beta}\,. \end{aligned} \tag{2.61}$$

It is not difficult to verify that these components satisfy the condition:

$$g_{ik}g^{kj} = \begin{pmatrix} 1 + O(1/c^6) & 0 + O(1/c^5) \\ 0 + O(1/c^5) & \delta_{\alpha\beta} + O(1/c^4) \end{pmatrix}. \tag{2.62}$$

The metric components (2.60) demonstrate that in this approximation the spacial section $t = const$ is conformal to the Euclidean metric. Hence, the spacial coordinates are isotropic ones. It is convenient to use three-dimensional spherical coordinates

r_3, θ, ψ instead of the Cartesian coordinates $x^1 \equiv x, x^2 \equiv y, x^3 \equiv z$. In these coordinates the metric (2.1) reads:

$$ds^2 \approx \left(1 - \frac{r_g}{r_3} + \frac{r_g^2}{2r_3^2} - \frac{Dv^2}{2(D-2)c^2}\frac{r_g}{r_3}\right) c^2 dt^2$$
$$+ \frac{2(D-1)}{(D-2)c}\frac{r_g}{r_3} cdt \sum_{\alpha=4}^{D} v_\alpha dx^\alpha$$
$$- \left(1 + \frac{1}{D-2}\frac{r_g}{r_3}\right)\left(dr_3^2 + r_3^2 d\theta^2 + r_3^2 \sin^2\theta d\psi^2\right)$$
$$- \left(1 + \frac{1}{D-2}\frac{r_g}{r_3}\right)\left((dx^4)^2 + (dx^5)^2 + \ldots + (dx^D)^2\right) . \qquad (2.63)$$

As we mentioned above, this metric corresponds to a gravitating mass in the rest in our three dimensional space. If the mass is smeared over extra dimensions, the appropriate velocity components vanish.

3. Classical gravitational tests

Now, we want to check the obtained above multidimensional metric (2.63) from the point of its consistency with the famous classical tests: frequency shift, perihelion shift, deflection of light and time delay of radar echoes (the Shapiro time delay effect). We also want to calculate the parameterized post-Newtonian (PPN) parameters for obtained metric coefficients. It is well known that four-dimensional general relativity is in good agreement with these experiments and observed PPN parameters. Can the considered Kaluza-Klein models with point-like sources also be in concordance with observations?

3.1. Frequency shift

To investigate the gravitational redshift formula in the spacetime (2.63), we can use the famous expression for relation between the frequency ω_1 of a light signal emitted at a point 1 with the metric component $g_{00}|_1$ and the frequency ω_2 received at a point 2 with the metric component $g_{00}|_2$:

$$\omega_1 \left[(g_{00})^{1/2}\right]_1 = \omega_2 \left[(g_{00})^{1/2}\right]_2 . \qquad (3.1)$$

Therefore, up to the order $1/c^2$ we get

$$\omega_2 \approx \omega_1 \left(1 + \frac{\varphi_1 - \varphi_2}{c^2}\right) , \qquad (3.2)$$

where non-relativistic potential φ is given by (2.57). In considered approximation, this formula exactly coincides with the one from general relativity. Therefore, for this type of experiments it is hardly possible to observe the difference between the usual four-dimensional general relativity and multidimensional Kaluza-Klein models.

3.2. Perihelion shift

Let us consider now the motion of a test body of mass m' in the gravitational field described by metric (2.63). The Hamilton-Jacobi equation

$$g^{ik}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^k} - m'^2 c^2 = 0 \qquad (3.3)$$

for this test body moving in the orbital plane $\theta = \pi/2$ reads

$$\frac{1}{c^2}\left(1+\frac{r_g}{r_3}+\frac{r_g^2}{2r_3^2}+\frac{Dv^2}{2(D-2)c^2}\frac{r_g}{r_3}\right)\left(\frac{\partial S}{\partial t}\right)^2-\frac{2(D-1)v^\alpha}{(D-2)c^2}\frac{r_g}{r_3}\frac{\partial S}{\partial t}\frac{\partial S}{\partial x^\alpha}$$
$$-\left(1-\frac{1}{D-2}\frac{r_g}{r_3}\right)\left(\frac{\partial S}{\partial r_3}\right)^2-\frac{1}{r_3^2}\left(1-\frac{1}{D-2}\frac{r_g}{r_3}\right)\left(\frac{\partial S}{\partial \psi}\right)^2$$
$$-\left(1-\frac{1}{D-2}\frac{r_g}{r_3}\right)\left[\left(\frac{\partial S}{\partial x^4}\right)^2+...+\left(\frac{\partial S}{\partial x^D}\right)^2\right]-m'^2c^2\approx 0\,. \tag{3.4}$$

We investigate this equation by separation of variables considering the action in the form

$$S=-E't+M\psi+S_{r_3}(r_3)+S_4\left(x^4\right)+...+S_D\left(x^D\right)\,. \tag{3.5}$$

Here, $E' \approx m'c^2 + E$ is the energy of the test body, which includes the rest energy $m'c^2$ and non-relativistic energy E, and M is the angular momentum. Substituting this expression for the action S in the formula (3.4), we obtain an expression for $(dS_{r_3}/dr_3)^2$ holding there the terms up to the order $1/c^2$:

$$\left(\frac{dS_{r_3}}{dr_3}\right)^2\approx\frac{E'^2}{c^2}\left(1-\frac{1}{D-2}\frac{r_g}{r_3}\right)^{-1}\left(1+\frac{r_g}{r_3}+\frac{r_g^2}{2r_3^2}+\frac{Dv^2}{2(D-2)c^2}\frac{r_g}{r_3}\right)$$
$$-\frac{M^2}{r_3^2}+E'\left(1-\frac{1}{D-2}\frac{r_g}{r_3}\right)^{-1}\frac{2(D-1)v^\alpha}{(D-2)c^2}\frac{r_g}{r_3}\frac{\partial S}{\partial x^\alpha}$$
$$-\left(\frac{dS_4}{dx^4}\right)^2-\ldots-\left(\frac{dS_D}{dx^D}\right)^2-m'^2c^2\left(1-\frac{1}{D-2}\frac{r_g}{r_3}\right)^{-1}$$
$$\approx\left(2m'E-\left(p_4^2+...+p_D^2\right)+\frac{E^2}{c^2}\right)-\frac{1}{r_3^2}\left(M^2-\frac{Dm'^2c^2r_g^2}{2(D-2)}\right) \tag{3.6}$$
$$+\frac{1}{r_3}\left(m'^2c^2r_g+\frac{2(D-1)}{D-2}m'Er_g+\frac{D}{2(D-2)}m'^2r_gv^2+\frac{2(D-1)}{(D-2)}m'r_g\sum_{\alpha=4}^{D}v^\alpha p_\alpha\right)\,,$$

where $p_\alpha = \partial S/\partial x^\alpha = dS_\alpha/dx^\alpha$ $(\alpha = 4, \ldots, D)$ are the components of momentum of the test body in the extra dimensions. If the gravitating and test masses are localized on the same brane then these components are equal to zero. Integrating the square root of this expression with respect to r_3, we get S_{r_3} in the following form:

$$S_{r_3}\approx\int\left[\left(2m'E-\left(p_4^2+...+p_D^2\right)+\frac{E^2}{c^2}\right)\right.$$
$$+\frac{1}{r_3}\left(m'^2c^2r_g+\frac{2(D-1)}{D-2}m'Er_g+\frac{D}{2(D-2)}m'^2r_gv^2+\frac{2(D-1)}{(D-2)}m'r_g\sum_{\alpha=4}^{D}v^\alpha p_\alpha\right)$$
$$\left.-\frac{1}{r_3^2}\left(M^2-\frac{Dm'^2c^2r_g^2}{2(D-2)}\right)\right]^{1/2}dr_3\,, \tag{3.7}$$

It is well known (see e.g. § 47 in [25]) that for any integral of motion I of a system with action S the following equation should hold:

$$\frac{\partial S}{\partial I}=const\,. \tag{3.8}$$

Because the angular momentum M is the integral of motion, the trajectory of the test body is defined by the equation

$$\frac{\partial S}{\partial M} = \psi + \frac{\partial S_{r_3}}{\partial M} = \text{const}\,, \tag{3.9}$$

where we use (3.5).

Let now the Sun be the gravitating mass and the planets of the solar system be the test bodies. Then, the change of the angle during one revolution of a planet on an orbit is

$$\Delta\psi = -\frac{\partial}{\partial M}\Delta S_{r_3}\,, \tag{3.10}$$

where ΔS_{r_3} is the corresponding change of S_{r_3}. It is well known that the perihelion shift originates due to small relativistic correction ε to M^2 in S_{r_3}: $M^2/r_3^2 \Rightarrow (M^2-\varepsilon)/r_3^2$. (3.7) shows that in our case $\varepsilon = Dm'^2c^2r_g^2/[2(D-2)]$. Expanding S_{r_3} in powers of this correction:

$$\begin{aligned} S_{r_3} &= S_{r_3}(M^2-\varepsilon) \approx S_{r_3}^{(0)} - \varepsilon\frac{\partial S_{r_3}^{(0)}}{\partial M^2} \\ &= S_{r_3}^{(0)} - \frac{\varepsilon}{2M}\frac{\partial S_{r_3}^{(0)}}{\partial M} = S_{r_3}^{(0)} - \frac{Dm'^2c^2r_g^2}{4(D-2)M}\frac{\partial S_{r_3}^{(0)}}{\partial M}\,, \end{aligned} \tag{3.11}$$

where $S_{r_3}^{(0)} \equiv S_{r_3}(M^2)$, we obtain

$$\Delta S_{r_3} \approx \Delta S_{r_3}^{(0)} - \frac{Dm'^2c^2r_g^2}{4(D-2)M}\frac{\partial \Delta S_{r_3}^{(0)}}{\partial M}\,. \tag{3.12}$$

Differentiating this equation with respect to M we get

$$\Delta\psi \approx 2\pi + \frac{D\pi m'^2c^2r_g^2}{2(D-2)M^2}\,, \tag{3.13}$$

where we took into account $-\partial\Delta S_{r_3}^{(0)}/\partial M = \Delta\psi^{(0)} = 2\pi$. Therefore, the second term in (3.13) gives the required formula for the perihelion shift in our multidimensional case:

$$\delta\psi = \frac{D\pi m'^2c^2r_g^2}{2(D-2)M^2} = \frac{D\pi r_g}{(D-2)a(1-e^2)}\,, \tag{3.14}$$

where in this equation we used the well-known relation $M^2 = m'^2r_gc^2a(1-e^2)/2$ with a and e being the semi-major axis and the eccentricity of the ellipse, respectively. For the three-dimensional case $D=3$, this equation exactly coincides with formula (101.7) in [21]. It can be easily seen that the result (3.14) does not depend on motion of the gravitating and test masses in the extra dimensions.

It make sense to apply this formula to Mercury because in the solar system it has the most significant discrepancy between the measurement value of the perihelion shift and its calculated value using Newton's formalism. The observed discrepancy is 43.11 ± 0.21 arcsec per century. This missing value is usually explained by the relativistic effects of the form of (3.14). However, only in three-dimensional case $D=3$ (3.14) gives the satisfactory result $42.94''$ which is within the measurement accuracy. For $D=4$ and $D=9$ models we obtain $28.63''$ and $18.40''$, respectively, which are very far from the observable value.

3.3. Deflection of light

Let us consider now the propagation of light in gravitational field with metric (2.63). In the case of massless particles, the Hamilton-Jacobi equation (3.3) is reduced to the eikonal equation:

$$g^{ik}\frac{\partial\Psi}{\partial x^i}\frac{\partial\Psi}{\partial x^k}=0\,, \tag{3.15}$$

which for the metric (2.63) reads

$$\begin{aligned}&\frac{1}{c^2}\left(1+\frac{r_g}{r_3}+\frac{r_g^2}{2r_3^2}+\frac{Dv^2}{2(D-2)c^2}\frac{r_g}{r_3}\right)\left(\frac{\partial\Psi}{\partial t}\right)^2-\frac{2(D-1)v^\alpha}{(D-2)c^2}\frac{r_g}{r_3}\frac{\partial\Psi}{\partial t}\frac{\partial\Psi}{\partial x^\alpha}\\&-\left(1-\frac{1}{D-2}\frac{r_g}{r_3}\right)\left(\frac{\partial\Psi}{\partial r_3}\right)^2-\frac{1}{r_3^2}\left(1-\frac{1}{D-2}\frac{r_g}{r_3}\right)\left(\frac{\partial\Psi}{\partial\psi}\right)^2\\&-\left(1-\frac{1}{D-2}\frac{r_g}{r_3}\right)\left[\left(\frac{\partial\Psi}{\partial x^4}\right)^2+\ldots+\left(\frac{\partial\Psi}{\partial x^D}\right)^2\right]\approx 0\,,\end{aligned} \tag{3.16}$$

where we take into account that light propagates in the orbital plane $\theta=\pi/2$. The eikonal function Ψ can be written in the form

$$\Psi=-\omega_0 t+\frac{\rho\omega_0}{c}\psi+\Psi_{r_3}(r_3)+\Psi_4\left(x^4\right)+\Psi_5\left(x^5\right)+\ldots+\Psi_D\left(x^D\right)\,, \tag{3.17}$$

where $\omega_0=-\partial\Psi/\partial t$ is the frequency of light and ρ is a constant. Later we will show that ρ is the impact parameter, i.e. distance of closest approach of the ray's path to the gravitating mass. Taking into account that $k=\omega_0/c$ is the absolute value of the wave-vector, it is clear that $M\equiv\rho k=\rho\omega_0/c$ plays the role of the angular momentum for the light beam.

Now we consider the natural case when the light propagates in our three-dimensional space and does not have components of momentum in the extra dimensions, that is $p_\alpha=d\Psi_\alpha/dx^\alpha\equiv 0,\quad \alpha=4,\ldots,D$. Then from (3.16), using (3.17), we obtain up to the order $O(1/c^4)$ the following formula:

$$\begin{aligned}\left(\frac{d\Psi_{r_3}}{dr_3}\right)^2&\approx\frac{\omega_0^2}{c^2}\left(1-\frac{1}{D-2}\frac{r_g}{r_3}\right)^{-1}\left(1+\frac{r_g}{r_3}+\frac{r_g^2}{2r_3^2}+\frac{Dv^2}{2(D-2)c^2}\frac{r_g}{r_3}\right)-\frac{\rho^2\omega_0^2}{c^2r_3^2}\\&\approx\frac{\omega_0^2}{c^2}\left(1+\frac{D-1}{D-2}\frac{r_g}{r_3}-\frac{\rho^2}{r_3^2}\right)\,.\end{aligned} \tag{3.18}$$

Integrating this expression we get:

$$\Psi_{r_3}\approx\frac{\omega_0}{c}\int\left(1+\frac{D-1}{D-2}\frac{r_g}{r_3}-\frac{\rho^2}{r_3^2}\right)^{1/2}dr_3\,. \tag{3.19}$$

Considering the term with r_g/r_3 as a small relativistic correction, we expand the integrand up to the order $O(1/c^3)$:

$$\begin{aligned}\Psi_{r_3}&\approx\Psi_{r_3}^{(0)}+\frac{D-1}{2(D-2)}\frac{r_g\omega_0}{c}\int\left(r_3^2-\rho^2\right)^{-1/2}dr_3\\&=\Psi_{r_3}^{(0)}+\frac{D-1}{2(D-2)}\frac{r_g\omega_0}{c}\operatorname{arccosh}\frac{r_3}{\rho}\,,\end{aligned} \tag{3.20}$$

where the non-relativistic (i.e. gravity is absent: $r_g\equiv 0$) eikonal function is

$$\Psi_{r_3}^{(0)}=\frac{\omega_0}{c}\int\left(1-\frac{\rho^2}{r_3^2}\right)^{1/2}dr_3\equiv\int\left(\left(\frac{\omega_0}{c}\right)^2-\frac{M^2}{r_3^2}\right)^{1/2}dr_3\,. \tag{3.21}$$

For this non-relativistic approximation the trajectory of the light beam is a straight line. Indeed, in this case (by full analogy with (3.9)) we have

$$\frac{\partial \Psi^{(0)}}{\partial M} = \psi^{(0)} + \frac{\partial \Psi^{(0)}_{r_3}}{\partial M} = \psi^{(0)} - \arccos(\rho/r_3) = 0\,, \tag{3.22}$$

where the constant is taken in such a way that $\psi^{(0)} \to \pi/2$ for $r_3 \to \infty$. Thus, the trajectory $\rho = r_3 \cos\psi^{(0)}$ is the straight line. Obviously, in the non-relativistic case the total change of the angle $\psi^{(0)}$ is $\Delta\psi^{(0)} = -\partial\Delta\Psi^{(0)}_{r_3}/\partial M = \pi$.

Coming back to the relativistic case (3.20), for the light beam travelling from some distance $r_3 = R$ to the closest approach to the gravitating mass at $r_3 = \rho$ and again to the distance $r_3 = R$, the change of the eikonal function is

$$\Delta\Psi_{r_3} \approx \Delta\Psi^{(0)}_{r_3} + \frac{D-1}{D-2}\frac{r_g\omega_0}{c}\mathrm{arccosh}\frac{R}{\rho}\,. \tag{3.23}$$

The corresponding change of the polar angle ψ is

$$\frac{\partial\Psi}{\partial M} = \psi + \frac{\partial\Psi_{r_3}}{\partial M} = \mathrm{const}$$

$$\Longrightarrow \quad \Delta\psi = -\frac{\partial\Delta\Psi_{r_3}}{\partial M} \approx -\frac{\partial\Delta\Psi^{(0)}_{r_3}}{\partial M} + \frac{D-1}{D-2}\frac{r_g R}{\rho}\left(R^2-\rho^2\right)^{-1/2}\,. \tag{3.24}$$

Thus in the limit $R \to +\infty$ we finally get:

$$\Delta\psi \approx \pi + \frac{D-1}{D-2}\frac{r_g}{\rho}\,. \tag{3.25}$$

Therefore, the second term in (3.25) gives the required formula for the deflection of light in our multidimensional case:

$$\delta\psi = \frac{D-1}{D-2}\frac{r_g}{\rho} \tag{3.26}$$

For the three-dimensional case $D = 3$, this equation exactly coincides with formula (101.9) in [21].

Now we apply this formula to the Sun. Obviously, the radius R_{Sun} of the Sun is much greater than the size of the extra dimensions and approximation (2.57) works well on the distances $r_3 \geq R_{Sun}$. For general relativity and for a ray that grazes the Sun's limb (i.e. $\rho \approx R_{Sun}$) $\delta\psi \approx 1.75$ arcsec which is in very good agreement with observational data [12]. (3.26) shows that we get this value of $\delta\psi$ only for usual three-dimensional space. In the case $D = 4$ and $D = 9$ we obtain correspondingly $\delta\psi \approx 1.31^{''}$ and $\delta\psi \approx 1.00^{''}$ which are very far from the observable value.

3.4. Parameterized post-Newtonian Parameters and gravitational tests

It is well known (see e.g. [26, 27]) that in PPN formalism the static, spherically symmetric metric in isotropic coordinates reads

$$ds^2 = \left(1 - \frac{r_g}{r_3} + \beta\frac{r_g^2}{2r_3^2}\right)c^2dt^2 - \left(1 + \gamma\frac{r_g}{r_3}\right)\sum_{i=1}^{3}\left(dx^i\right)^2\,. \tag{3.27}$$

In general relativity, $\beta = \gamma = 1$. However, simple comparison of equations (3.27) and (2.63) shows that the PPN parameters β and γ in our case are

$$\beta = 1\,, \qquad \gamma = \frac{1}{D-2}\,. \tag{3.28}$$

The latter expression shows that parameter γ coincides with the corresponding value in general relativity if $D = 3$. Only in this case $\gamma = 1$. According to the experimental data, γ should be very close to 1. The tightest constraint on γ comes from the Shapiro time-delay experiment using the Cassini spacecraft: $\gamma - 1 = (2.1 \pm 2.3) \times 10^{-5}$ [13, 28, 29]. On the other hand, for $D = 4, 9$ we get from (3.28) that $\gamma - 1 = -1/2, -6/7$ respectively, which is very far from the experimental data.

The formulas of the gravitational tests can be expressed via the PPN parameters [26, 28]. For example, the perihelion shift and the deflection of light read correspondingly:

$$\delta\psi = \frac{1}{3}\,(2 + 2\gamma - \beta)\;\frac{3\pi r_g}{a(1-e^2)}\,, \tag{3.29}$$

$$\delta\psi = \;(1+\gamma)\;\frac{r_g}{\rho}\,. \tag{3.30}$$

Now, if we substitute in these expressions the values from equation (3.28) for β and γ, then we exactly restore our formulas (3.14) and (3.26).

It makes sense to present the expression for the *time delay of radar echoes* (the Shapiro time delay effect) via the PPN parameters. This effect consists in time difference of propagation of electromagnetic signals between two points (or for a round trip) in the curved and flat spaces. Usually, a signal transmits from the Earth through a region near the Sun to another planet or satellite and then reflects back to the Earth. If the planet (or satellite) is on the far side of the Sun from the Earth (superior conjunction), then the formula for the time delay reads [26, 28]:

$$\delta t = (1+\gamma)\frac{r_g}{c}\ln\left(\frac{4r_{Earth}\,r_{planet}}{R_{Sun}^2}\right)\,, \tag{3.31}$$

where r_{Earth} and r_{planet} are the distances from the Sun to the Earth and to the planet, respectively. If we put into this formula the value of γ from equation (3.28), we get

$$\delta t = \frac{D-1}{D-2}\frac{r_g}{c}\ln\left(\frac{4r_{Earth}\,r_{planet}}{R_{Sun}^2}\right)\,. \tag{3.32}$$

Obviously, this formula coincides with the general relativity only for $D = 3$ (see e.g. [27]). For all others values of D the time delay differs from the general relativity by the factor of $O(1)$.

4. Conclusion

In our paper we investigated classical gravitational tests (frequency shift, perihelion shift, deflection of light and time delay of radar echoes) for multidimensional models with compact internal spaces in the form of tori. We supposed that in the absence of gravitating masses the metric is a flat one. Gravitating point-like masses (moving or at rest) perturb this metric and we considered these perturbations in a weak field approximation. In this approximation, we obtained the asymptotic form of the metric coefficients. Until this point we did not require the compactness of the extra dimensions. This approach is valid for any number of spatial dimensions $D \geq 3$ and generalizes well-known calculations [21] in four-dimensional space-time. Then, we admitted that, first, the extra dimensions are compact and have the topology of tori and, second, gravitational potential far away from gravitating masses tends to non-relativistic Newtonian limit. It gave us a possibility to specify the non-relativistic gravitational potential for considered models. In turn, it enabled us to specify the

metric coefficients. In the case of a gravitating delta-shaped body at rest, we used these metric coefficients to calculate frequency shift, perihelion shift, deflection of light and parameterized post-Newtonian parameters β and γ. With the help of PPN parameter γ, we also obtain the formula for the time delay of radar echoes. We demonstrated that for the frequency shift type experiment it is hardly possible to observe the difference between the usual four-dimensional general relativity and multidimensional Kaluza-Klein models. However, the situation is quite different for perihelion shift, deflection of light and the Shapiro time delay effect. In these three cases we obtained formulas which generalize the corresponding ones in general relativity. We showed that all of these formulas depend on a total number of spatial dimensions D and they are in good agreement with observations only in ordinary three-dimensional space $D = 3$. This result does not depend explicitly on the size of the extra dimensions. Therefore, it is impossible to avoid the problem with classical gravitational tests in a limit of arbitrary small sizes of the extra dimensions.

Therefore, our results show that in considered multidimensional Kaluza-Klein models the point-like gravitating masses cannot produce gravitational field which corresponds to the classical gravitational tests. Moreover, it is not difficult to show (see our forthcoming paper), that similar problem arises in the case of a compact static spherically symmetric perfect fluid with the following conditions for the energy-momentum tensor: $T_{00} \gg T_{0\alpha}, T_{\alpha\beta}\, , \quad \alpha, \beta = 1, \ldots, D$. To avoid this problem, it is necessary to break a symmetry between our three usual spatial dimensions and the extra dimensions. The branes are among the most natural candidates for solving this problem. Our results work in favor of the brane-world models. It is of interest also to check models with a non-linear action $f(R)$. However, to prove viability of these models it is necessary to perform the similar investigations.

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